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Pseudo-reality and pseudo-adjointness of Hamiltonians

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Abstract
We define pseudo-reality and pseudo-adjointness of a Hamiltonian, H, as $\rho H \rho^{-1} = H^*$ and $\mu H \mu^{-1} = H'$, respectively. We prove that the former yields the *necessary* condition for a spectrum to be real whereas the latter helps in fixing a definition for the inner-product of the eigenstates. Here we separate out the adjointness of an operator from its Hermitian adjointness. It turns out that a Hamiltonian possessing a real spectrum is first pseudo-real, further it could be Hermitian, *PT*-symmetric or pseudo-Hermitian.

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If a Hamiltonian, H, having a discrete spectrum commutes with an anti-linear operator, X, the spectrum contains either real or complex conjugate pairs of eigenvalues [1], in the former case H and X have common eigenstates and in the latter case it does not happen. During the last few years a new scope for even a non-Hermitian Hamiltonian to possess a real spectrum can be seen in this light. It has been found [2] that a *PT*-invariant Hamiltonian, i.e. $[PT, H] = 0$, where $P : x \to -x, T : \mathrm{i} \to -\mathrm{i}$ has real eigenvalues when $X = PT$ and H admit common eigenstates and the *PT*-symmetry is called exact. Otherwise, the eigenvalues are complex conjugate pairs and *PT*-symmetry is said to be spontaneously broken. Very interestingly, prior to the calculations, hitherto just by looking at the Hamiltonian, one cannot tell whether a *PT*-symmetric potential will have real or complex (conjugate pairs of) energy eigenvalues. This intriguing feature has inspired the pursuit [2–7] of both analytically and numerically solved models of *PT*-symmetric potentials.

Further, the phenomenon of real eigenvalues of non-Hermitian Hamiltonians has been found to be connected with the already known concept of pseudo-Hermiticity. A Hamiltonian is pseudo-Hermitian [8–17] if

$$\eta H \eta^{-1} = H^{\dagger} \tag{1}$$

where η is called the metric and it is a linear operator. It has also been known that

$$(E_m^* - E_n)\Psi_m^{\dagger} \cdot \eta \Psi_n = 0 \qquad N_{\eta,n} = \Psi_n^{\dagger} \cdot \eta \Psi_n. \tag{2}$$

Here we propose to choose matrix notation for the subtle reason that this notation has a separate and explicit sign for the adjoint (transpose) operation. The sign † jointly denotes complex conjugation, *, and transpose (adjoint), $'$, of the operators or vectors. Note that (2) merely asserts two important features of the eigenstates: (i) if eigenvalues are real and distinct, the eigenstates will be $\boldsymbol{\eta}$-orthogonal as $\Psi_m^\dagger \cdot \boldsymbol{\eta}\Psi_n = \epsilon\delta_{m,n}$; (ii) complex eigenvalues will have zero pseudo-norm, i.e., $N_{\eta,n} = 0$. Here, we must bring home the fact that the concept of pseudo-Hermiticity as such does not yield an explicit proof for the reality of eigenvalues (even under any further condition), it can only support real eigenvalues indirectly (see equation (2)). This shortcoming of pseudo-Hermiticity which has gone unnoticed [8–17] both recently and initially motivates the present work.

Several *PT*-symmetric potentials having a real spectrum have been found to be parity pseudo-Hermitian where $\boldsymbol{\eta} = P$ [10]. Several complex potentials which are both *PT*-symmetric and non-*PT*-symmetric have been found to be pseudo-Hermitian when $\boldsymbol{\eta} = \mathrm{e}^{-\theta p_x}$ [10]. This operator affects an imaginary shift in the coordinate, i.e. $\boldsymbol{\eta}x\boldsymbol{\eta}^{-1} = x + \mathrm{i}\theta$. Several other Hamiltonians of both types have been reported [11, 12] to be pseudo-Hermitian under $\boldsymbol{\eta} = \mathrm{e}^{\phi(x)}$: a gauge-like transformation. (Weak) pseudo-Hermiticity [13], pseudoanti-Hermiticity [14] of Hamiltonians and a recipe [15] for construction of pseudo-Hermitian potentials have also been proposed. Further, pseudo-Hermitian random matrices have been conceived to propose Gaussian pseudo-unitary ensembles (GPUE) [16].

Since *PT*-symmetry can provide the contact with physical situations and systems, recasting of the more general property of pseudo-Hermiticity in terms of *PT*-symmetry has been taken up. This has been achieved mainly through the proposal of the interesting existence of anti-linear commutants [9, 13]. Later these commutants have been identified as generalized symmetries *C*, *PT* and *CPT* [18–24] of the non-Hermitian Hamiltonians possessing real eigenvalues. Though *C*, which denotes a novel charge-conjugation symmetry analogous to that well known in relativistic field theory, was first proposed [18] due to the characteristic indefiniteness of the *PT*-norm [7].

At this stage of the developments, we find that the adjointness of a Hamiltonian has not been taken into account when we discuss the *PT*-symmetry or pseudo-Hermiticity of a Hamiltonian. As a result, we find that a potential despite being both *PT*-symmetric and pseudo-Hermitian and possessing a real spectrum does not satisfy (e.g. [11]) the *PT*-orthogonality (*PT*-inner-product) [7]:

$$(E_m^* - E_n)\Psi_m^{PT} \cdot \Psi_n = 0 \qquad N_{PT,n} = \Psi_n^{PT'} \cdot \Psi_n. \tag{3}$$

It does, however, satisfy the $\boldsymbol{\eta}$-pseudo-orthogonality condition (2). This is as though *PT*-symmetry is not enough to ensure orthogonality of eigenstates. A special analysis has been carried out [17] to uphold *PT*-symmetry in this regard, eventually it yielded a condition more akin to (2).

Moreover, as mentioned above, the concept of pseudo-Hermiticity at best does not contradict the occurrence of the real eigenvalues, nevertheless it does not provide a direct proof for it. In this regard the following works are important. It has been proved that if a pseudo-Hermitian Hamiltonian possesses real eigenvalues then there exists (one can find) a metric of the type $\boldsymbol{\eta} = OO^\dagger$ [9] or $(OO^\dagger)^{-1}$ [13] under which the Hamiltonian is pseudo-Hermitian. Next, following matrix algebra it has been stated and proved [22] that if a matrix Hamiltonian has real eigenvalues and a diagonalizing matrix D then it is pseudo-Hermitian under $\boldsymbol{\eta}_+ = (DD^\dagger)^{-1}$ and vice versa. However, following these works one can only find that the metric $\boldsymbol{\eta}$ and also the anti-linear commutants [9, 13] are dynamical as they are essentially generated from the eigenbasis and so they are entangled with the Hamiltonian.

In this work, we introduce the concept of pseudo-reality and pseudo-adjointness of a Hamiltonian by proposing to separate out the adjointness of an operator from the Hermitian adjointness, a subtle point which has been missed in the developments described above.

Let us first discuss the adjointness of an operator. We propose to use $'$ sign for adjoint and transpose if the Hamiltonian is in differential and matrix form, respectively. The adjoint of a differential operator A denoted as A' is defined as [25]

$$u \cdot Av - v \cdot A'u = \frac{\mathrm{d}W(u, v)}{\mathrm{d}x} \tag{4}$$

i.e. the right-hand side is an exact differential and W is called the bilinear concomitant [25]. The functions u, v are two arbitrary vectors from a vector space. Here the dot denotes simple multiplication. Subsequently, we have

$$\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n}\right)' = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} \qquad n = 1, 2, \ldots. \tag{5}$$

Thus for the quantum mechanical operators—position, momentum and kinetic energy—we have

$$(x)' = x \quad (p_x)' = -p_x \quad \text{and} \quad (K)' = K. \tag{6}$$

Thus, Hamiltonians of the type $p_x^2\big/(2m) + V(x)$ are self-adjoint, i.e. $H = H'$. Usually, we use the concept of Hermitian adjointness in quantum mechanics, i.e.

$$(p_x)^\dagger = \left(-\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}x}\right)'^* = p_x \tag{7}$$

and call an operator $A \equiv p_x, K$ and x to be self-(Hermitian)adjoint by also noting that $\langle A\Psi|\Phi\rangle = \langle \Psi|A^\dagger \Phi\rangle$ [25, 26]. The phrase Hermitian is also dropped from the self-(Hermitian) adjoint and it is taken for granted in Hermitian quantum mechanics. Nevertheless, while investigating the real spectrum of non-Hermitian Hamiltonians, we have to disentangle these two. Note that, in matrix notation, we have

$$(Au)' \cdot v - u' \cdot A'v = 0 \tag{8}$$

if $'$ denotes the transpose of a matrix and a dot denotes matrix multiplication. In matrix algebra, incidentally one defines the 'adjoint' of a matrix as $\mathrm{adj}(A) = A^{-1}|A|$, which should be taken as a misnomer for quantum mechanical discussions. Let us keep in mind that $(p_x)^* = -p_x$ and the following transformations,

$$T p_x T^{-1} = -p_x = P p_x P^{-1} \qquad T x T^{-1} = x = P(-x)P^{-1} \qquad T K T^{-1} = K = P K P^{-1} \tag{9}$$

for further discussions.

We propose to call a Hamiltonian, H, pseudo-real, if

$$\rho H \rho^{-1} = H^* \tag{10a}$$

where ρ is such that

$$\rho^* \rho = 1 \tag{10b}$$

noting that $H^{**} = \rho^* H^* \rho^{-1*} \Rightarrow H = \rho^* \rho H \rho^{-1} \rho^{-1*} \Rightarrow \rho^* \rho = 1$. Next, we propose to call H as pseudo-adjoint, if

$$\mu H \mu^{-1} = H' \tag{11a}$$

where

$$\mu = \mu' \tag{11b}$$

noting that $H'' = \mu^{-1\prime} H' \mu' \Rightarrow H = \mu^{-1\prime} \mu H \mu^{-1} \mu' \Rightarrow \mu' = \mu$.

Proposition 1. *If a pseudo-real (10) Hamiltonian, H, possesses a discrete spectrum, then the eigenvalues are either real or complex conjugate pairs.*

Proof. Recalling that $(AB)^* = A^*B^*$, let $H\Psi = E\Psi$

$$\begin{aligned} &\Rightarrow (H\Psi)^* = (E\Psi)^* \Rightarrow H^*\Psi^* = E^*\Psi^* \\ &\Rightarrow \rho H\rho^{-1}\Psi^* = E^*\Psi^* \Rightarrow H(\rho^{-1}\Psi^*) = E^*(\rho^{-1}\Psi^*). \end{aligned} \tag{12}$$

□

The first and the last parts imply that if E is an eigenvalue with eigenfunction, Ψ, then so is E^* with eigenfunction, $\rho^{-1}\Psi^*$. Since each Ψ gets mapped to $\rho^{-1}\Psi^*$, the multiplicities of eigenvalues will be the same.

Corollary. *If E is non-degenerate, then*

$$E = E^* \qquad \textit{if} \quad \rho^{-1}\Psi^* = \epsilon\Psi. \tag{13}$$

Proposition 2. *If a Hamiltonian, H, is pseudo-real (10) and pseudo-adjoint (11), then it is pseudo-Hermitian (1) under*

$$\eta = (\mu\rho^{-1})'. \tag{14}$$

Recall that $(AB)' = B'A'$.

Proof.

$$\rho H\rho^{-1} = \rho(\mu^{-1}H'\mu)\rho^{-1} \Rightarrow H^* = (\rho\mu^{-1}H'\mu\rho^{-1}) \Rightarrow H^{*\prime} = (\mu\rho^{-1})'H(\rho\mu^{-1})'. \tag{15}$$

□

Finally we have

$$(\mu\rho^{-1})'H((\mu\rho^{-1})')^{-1} = H^\dagger. \tag{16}$$

Further, the orthogonality of the eigenstates will follow according to (2), which now reads

$$(E_m^* - E_n)\Psi_m^\dagger \cdot (\mu\rho^{-1})'\Psi_n = 0 \qquad N_{\eta,n} = \Psi_n^\dagger \cdot (\mu\rho^{-1})'\Psi_n. \tag{17}$$

We must remark that ρ and μ like η are linear and non-unique.

Hermiticity of H follows when we have $\rho = \mu$. *PT*-symmetry of the Hamiltonian follows when we have $\rho = P$ and $\mu = \mathbf{1}$. In addition to this, if we treat complex conjugation as T in (13), we rediscover the fact that eigenvalues of a *PT*-symmetric potential will be real provided $PT\,\Psi = \epsilon\Psi$, i.e Ψ is also the eigenstate of *PT*. Let us have a quick illustration: if $H_0 = cp_x$, we find that this Hamiltonian is pseudo-real under parity P, it possesses real eigenvalues $\pm ck$ and the eigenstates are $\Psi = \mathrm{e}^{\pm ikx}$, with $\epsilon = 1$.

Proposition 3. *The ρ-pseudo-reality of a Hamiltonian, H, having a discrete spectrum is equivalent to the existence of an anti-linear commutant [13],* $\Theta = \rho^{-1}K_0$, *of H. Here* K_0 *denotes the operation of complex conjugation:* $K_0(AB + C) = A^*B^* + C^*$.

Proof.

$$[H, \Theta] = 0 \Leftrightarrow \rho^{-1}K_0H = H\rho^{-1}K_0 \Leftrightarrow \rho H\rho^{-1} = K_0HK_0^{-1} \Leftrightarrow \rho H\rho^{-1} = H^*. \tag{18}$$

□

Note that Θ unlike Θ_E [13] is not essentially eigenbasis dependent. Condition (13) for the reality of eigenvalues could be rewritten as

$$\Theta\Psi = \epsilon\Psi. \tag{19}$$

Thus, three hypotheses, namely, the simultaneity (19) of eigenstate Ψ for H and Θ, the commutativity of H and Θ (or equivalently the pseudo-reality of H) (18) and the reality of eigenvalues are mutually consistent as one finds that

$$[H, \Theta]\Psi = (E^* - E)\epsilon\Psi. \tag{20}$$

Given two, the third one is implied. The property (10b) renders that $\Theta^2 = 1$, ruling out the essential existence of Kramer's degeneracy when Hamiltonians are pseudo-real. It may be recalled that if the anti-linear commutant of H is such that $\Theta^2 = -1$, the energy levels are essentially doubly degenerate [26].

In the following, we review several available Hamiltonians (H_1–H_8) possessing a real discrete spectrum in the light of their pseudo-reality and pseudo-adjointness. The Hamiltonians of the type $H_1 = p_x^2\big/(2m)+V_e(x)+\mathrm{i}V_o(x)$ [2–7], where e and o denote even and odd functions, are such examples. For such PT-symmetric potentials, the self-adjointness of H is implied by $\boldsymbol{\mu} = \mathbf{1}$, and the following orthogonality condition

$$(E_m - E_n)\Psi'_m \cdot \Psi_n = 0 \tag{21}$$

will also work, automatically. Note the absence of † in (21). One can check that H_1 possesses real eigenvalues since it is pseudo-real, $PH_1P^{-1} = H_1^*$ and condition (13) is explicitly satisfied by the energy eigenstates. Several exactly solvable models of PT-symmetric potentials [2–7] are available for a verification.

The complex quasi-exactly solvable Hamiltonian [3]

$$H_2 = \frac{p_x^2}{2m} - (z\cosh 2x - 3\mathrm{i})^2 \tag{22}$$

has the first three eigenvalues (real if $z^2 \leqslant 1/4$) and eigenfunctions known analytically. H_2 was termed as PT-symmetric under $T : \mathrm{i} \to -\mathrm{i}$, and $P : x \to \mathrm{i}\pi/2 - x$. Note that both the operations do not commute [6]. We find that H_2 more appropriately is pseudo-real under the transformation $\boldsymbol{\rho} : x \to (\mathrm{i}\pi/2 - x)$ and self-adjoint ($\boldsymbol{\mu} = \mathbf{1}$). The eigenfunctions [3] can be checked to satisfy the proposed condition (13).

Let us consider the following Hamiltonian

$$H_3 = \frac{[p_x + \mathrm{i}\beta x]^2}{2m} + \frac{1}{2}m\alpha^2 x^2 \tag{23}$$

which admits real eigenvalues and real eigenvectors [11]. We find that H_3 is trivially pseudo-real (10) under $\boldsymbol{\rho} = \mathbf{1}$ and we will have real eigenvalues and real eigenfunctions too [11]. Next, H_3 is pseudo-adjoint (11) as $\mathrm{e}^{-\beta x^2} H_3\, \mathrm{e}^{\beta x^2} = H_3'$. So we have $\boldsymbol{\mu} = \mathrm{e}^{-\beta x^2} = \boldsymbol{\eta}$. Alternatively, we may take H_3 to be pseudo-real under $\boldsymbol{\rho} = P$ and then $\boldsymbol{\eta} = \mathrm{e}^{-\beta x^2} P$, also see [15]. Obviously, in both cases H_3 would rather be categorized as pseudo-Hermitian despite being PT-symmetric.

Next let us consider the Hermitian Hamiltonian

$$H_4 = \frac{[p_x - 3\gamma x^2]^2}{(2m)} + \frac{1}{2}m\alpha^2 x^2 \tag{24}$$

which has real eigenvalues. One can readily check that $\boldsymbol{\rho} = \boldsymbol{\mu} = P$, this leads to Hermiticity. We find that $\mathrm{e}^{-2\mathrm{i}\gamma x^3} H_4\, \mathrm{e}^{2\mathrm{i}\gamma x^3} = H_4^* = H_4'$ that means we again have the situation of Hermiticity where $\boldsymbol{\rho} = \boldsymbol{\mu} = \mathrm{e}^{-2\mathrm{i}\gamma x^3}$. Other interesting options are to choose $\boldsymbol{\rho} = P$ and $\boldsymbol{\mu} = \mathrm{e}^{-2\mathrm{i}\gamma x^3}$ or $\boldsymbol{\rho} = \mathrm{e}^{-2\mathrm{i}\gamma x^2}$ and $\boldsymbol{\mu} = P$. In both situations, we have $\boldsymbol{\eta} = \mathrm{e}^{-2\mathrm{i}\gamma x^3} P$, see also [17]. The $\boldsymbol{\eta}$-norm (2) will be indefinite, $(-1)^n$, $n = 0, 1, 2, \ldots$.

Intriguingly, when we choose to see even a Hermitian Hamiltonian (e.g. (24)) as *PT*-symmetric or pseudo-Hermitian both the norms are indefinite (positive–negative). However, the Hermitian norm, namely, $\Psi^\dagger\Psi$ remains definite (positive). This point has earlier been revealed and remarked [20, 22], however, it is often overlooked (see, e.g., [21, 23]).

Complex Morse (CM) potential $V^{\mathrm{CM}}(x) = (A + \mathrm{i}B)^2\,\mathrm{e}^{-2x} - (2C+1)(A+\mathrm{i}B)\,\mathrm{e}^{-x}$ which is non-*PT*-symmetric was found [6] to have real eigenvalues. Note that the real Morse (RM) potential is written as $V^{\mathrm{RM}}(x) = D^2\,\mathrm{e}^{-2x} - (2C+1)D\,\mathrm{e}^{-x}$ and $V^{\mathrm{CM}}(x)$ is nothing but $V^{\mathrm{RM}}(x - \mathrm{i}a)$. The Hamiltonian with this potential has been investigated [10] to be pseudo-Hermitian under $\boldsymbol{\eta} = \mathrm{e}^{-2ap_x}$. If real potentials $V(x)$ admit real eigenvalues then the potentials $V(x - \mathrm{i}a)$ are also found to possess identical eigenvalues. When real and imaginary parts of $V(x - \mathrm{i}a)$ are separated out, the rewritten potential would actually appear to be 'different' and even 'unrelated' to $V(x)$. The equivalence of two spectra will be due to the fact that the Hamiltonian $H(x) = p_x^2\big/(2m) + V(x)$ follows: $\mathrm{e}^{-ap_x}H(x - \mathrm{i}a)\,\mathrm{e}^{ap_x} = H(x)$. We find that $\mathrm{e}^{-2ap_x}H(x - \mathrm{i}a)\,\mathrm{e}^{2ap_x} = H(x + \mathrm{i}a)$ implying that $\boldsymbol{\rho} = \mathrm{e}^{-2ap_x}$ and $\boldsymbol{\mu} = \mathbf{1}$. Thus, both the orthogonality conditions (17) and (21) will be satisfied. We have indefinite norms: $N_{PT,n} = (-1)^n = N_{\eta,n}$.

In the following, we take up examples of simple pseudo-Hermitian matrices, for further demonstration of the pseudo-reality and pseudo-adjointness of Hamiltonians:

$$\begin{gathered} H_5 = \begin{bmatrix} a+\mathrm{i}b & c \\ c & a-\mathrm{i}b \end{bmatrix} \qquad H_6 = \begin{bmatrix} a+c & \mathrm{i}b \\ \mathrm{i}b & a-c \end{bmatrix} \\ H_7 = \begin{bmatrix} a & \mathrm{i}(b-c) \\ \mathrm{i}(b+c) & a \end{bmatrix} \qquad c^2 > b^2 . \end{gathered} \tag{25}$$

The eigenvalues of these matrices are $a \pm \sqrt{c^2 - b^2}$. In the following, we make an interesting use of Pauli matrices. For H_5, we find that $\boldsymbol{\rho} = \sigma_x$, $\boldsymbol{\mu} = \mathbf{1}$, so H_5 is pseudo-Hermitian under $\boldsymbol{\eta} = \sigma_x$. One can check that H_6 is pseudo-real under $\boldsymbol{\rho} = \sigma_z$ and $H_6 = H_6'$, so it is pseudo-Hermitian under $\boldsymbol{\eta} = \sigma_z$ as we have $\boldsymbol{\mu} = \mathbf{1}$ again. The Hamiltonian H_7 is pseudo-adjoint under σ_x and it is pseudo-real under σ_z to display pseudo-Hermiticity under $\boldsymbol{\eta} = \sigma_y$.

So far we could get $\boldsymbol{\rho}$ and $\boldsymbol{\mu}$ and hence $\boldsymbol{\eta}$ merely by inspection for several models of non-Hermitian Hamiltonian possessing real eigenvalues. Now we intend to show that there exists at least eigenbasis-dependent $\boldsymbol{\rho}$ and $\boldsymbol{\mu}$ when a non-Hermitian H possesses real eigenvalues. Let us define a real diagonal matrix $\mathcal{E} = \mathrm{diag}[E_1, E_2, E_3, \ldots, E_n]$, i.e., $\mathcal{E}^* = \mathcal{E}$ and $\mathcal{E}' = \mathcal{E}$.

Proposition 4. *If a complex Hamiltonian, H, possessing a real spectrum is diagonalizable by an operator D, it is pseudo-real (10) under $\boldsymbol{\rho} = D^*D^{-1}$ (the converse is also true).*

Proof.

$$D^{-1}HD = \mathcal{E} \Rightarrow D^{-1*}H^*D^* = \mathcal{E}^* \Rightarrow D^{-1*}\rho H\rho^{-1}D^* = \mathcal{E} \Rightarrow \boldsymbol{\rho} = D^*D^{-1}. \tag{26}$$

□

It may be checked that the interesting property of $\boldsymbol{\rho}$ in (10*b*) will be satisfied here. In the theory of matrices such a matrix is called circular.

Proposition 5. *If a Hamiltonian is diagonalizable by an operator D, it is pseudo-adjoint (11) under $\boldsymbol{\mu} = (DD')^{-1}$.*

Proof.

$$D^{-1}HD = \mathcal{E} \Rightarrow D'H'D^{-1\prime} = \mathcal{E}' \Rightarrow D'\mu H\mu^{-1}D^{-1\prime} = \mathcal{E} \Rightarrow \mu = (DD')^{-1}. \tag{27}$$

□

One may check that μ is self-adjoint (11*b*).

Proposition 6. *If a Hamiltonian H possessing a real spectrum is pseudo-real under* $\rho = D^*D^{-1}$ *and pseudo-adjoint under* $\mu = (DD')^{-1}$, *it is pseudo-Hermitian under* $\eta = (DD^\dagger)^{-1}$ *(the converse is also true).*

The proof follows straight from proposition 2. When H is Hermitian D will be unitary ($U^\dagger = U^{-1}$). We find that $\rho = U^*U^\dagger = \mu$ and $\eta = \mathbf{1}$.

Illustration. The following Hamiltonian H_8

$$H_8 = \begin{bmatrix} a+\mathrm{i}b & c+\mathrm{i}d \\ c-\mathrm{i}d & a-\mathrm{i}b \end{bmatrix} \qquad \Psi_1 = \begin{bmatrix} -\mathrm{e}^{-\mathrm{i}\theta} \\ \mathrm{e}^{-\mathrm{i}\phi} \end{bmatrix}$$
$$\Psi_2 = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\theta} \\ \mathrm{e}^{-\mathrm{i}\phi} \end{bmatrix} \qquad \Phi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \Phi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{28}$$

is pseudo-real under σ_x and possesses real eigenvalues $a \mp e$, where $e = \sqrt{c^2 + d^2 - b^2}$ if $c^2 + d^2 > b^2$. Here Ψ_n are eigenvectors of H and Φ_n provides a fundamental orthonormal basis. D can be constructed as $D = \sum_n \Psi_n \cdot \Phi_n'$. We find the expressions for ρ, μ and η_+,

$$\rho = \begin{bmatrix} 1 & -2\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}\sin\theta \\ 0 & \mathrm{e}^{2\mathrm{i}\phi} \end{bmatrix} \qquad \mu = \frac{\sec^2\theta}{2}\begin{bmatrix} 1 & -\mathrm{i}\,\mathrm{e}^{\mathrm{i}\phi}\sin\theta \\ -\mathrm{i}\sin\theta\,\mathrm{e}^{\mathrm{i}\phi} & \cos 2\theta\mathrm{e}^{2\mathrm{i}\phi} \end{bmatrix}$$
$$\eta_+ = \frac{\sec^2\theta}{2}\begin{bmatrix} 1 & -\mathrm{i}\sin\theta\,\mathrm{e}^{\mathrm{i}\phi} \\ \mathrm{i}\sin\theta\,\mathrm{e}^{-\mathrm{i}\phi} & 1 \end{bmatrix}. \tag{29}$$

We have introduced $\theta = \tan^{-1}(b/e)$ and $\phi = \tan^{-1}(d/c)$. This illustration also displays the non-uniqueness of ρ. Using $\rho = \sigma_x$ and μ as in (29), we can construct $\eta = (\mu\sigma_x)'$. This metric η will satisfy the orthogonality condition (2), however, it does not yield the η-norm (2) of the vectors Ψ_n as real, whereas the η_+-norm will be real and positive definite.

The *PT*-symmetric potentials in finite basis space yield finite-dimensional matrix Hamiltonians. In this regard, it is interesting to note that two-dimensional and three-dimensional matrix Hamiltonians obtained [27] for the potentials of type $V(x) = \mathrm{i}x^{2n+1}$ are pseudo-real where $\rho = \sigma_z$ and

$$\rho = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{30}$$

respectively. Some more interesting aspects of finite, *D*-dimensional, *PT*-symmetric Hamiltonians have recently been discussed [23, 24].

The separation of adjointness from Hermitian adjointness is natural when one deals with real eigenvalues of a non-Hermitian Hamiltonian. We conclude that this results in the condition of pseudo-reality which turns out to be the most elementary condition on a Hamiltonian for the eigenvalues to be real or complex-conjugate paris. Consequently, Hermitian, *PT*-symmetric, pseudo-Hermitian and weakly pseudo-Hermitian Hamiltonians are first pseudo-real. Next, the condition of pseudo-adjointness (11) helps in fixing the appropriate inner-product of the eigenstates. This is where the present work can be seen to propose a fundamental decomposition of η as $(\mu\rho^{-1})'$.

Eventually, we find that pseudo-reality comes to its logical end, that is, η-pseudo-Hermiticity, however, not without enriching and supplementing it with a relaxed *necessary* condition (10) and a crucial auxiliary condition (13) on the eigenstates for real eigenvalues. However, once again the condition for the reality of eigenvalues of pseudo-real Hamiltonians given in (13) or (19) and (26) is eigenbasis dependent and could not be liberated. In this regard, the simple examples presented here, where ρ and μ are independent of eigenbasis, are worthwhile. Admittedly, it is still not possible to predict, by merely looking at a pseudo-real Hamiltonian, whether it will have real eigenvalues.

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